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Selfadjoint operators in S-spaces

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Abstract

We study S-spaces and operators therein. An S-space is a Hilbert space $(\mathfrak{S}, (\cdot\,,-))$ with an additional inner product given by $[\cdot\,,-] := (U\cdot\,,-)$, where U is a unitary operator in $(\mathfrak{S}, (\cdot\,,-))$. We investigate spectral properties of selfadjoint operators in S-spaces. We show that their spectrum is symmetric with respect to the real axis. As a main result we prove that for each selfadjoint operator A in an S-space we find an inner product which turns $\mathfrak{S}$ into a Krein space and A into a selfadjoint operator therein. As a consequence we get a new simple condition for the existence of invariant subspaces of selfadjoint operators in Krein spaces, which provides a different insight into this well-know and in general unsolved problem.

Keywords: S-space; Krein space; Indefinite inner products; Selfadjoint operators; Invariant subspaces

1. Introduction

A complex linear space $\mathcal{H}$ with a Hermitian sesquilinear form $[\cdot\,,-]$ is called a *Krein space* if there exists a fundamental decomposition

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \tag{1}$$

with subspaces $\mathcal{H}_\pm$ being orthogonal to each other with respect to $[\cdot\,,-]$ such that $(\mathcal{H}_\pm, \pm[\cdot\,,-])$ are Hilbert spaces. If $\mathcal{H}_-$ or $\mathcal{H}_+$ is finite dimensional, then $(\mathcal{H}, [\cdot\,,-])$ is called a *Pontryagin*

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space. To each decomposition (1) there correspond a Hilbert space inner product $(\cdot,-)$ and a selfadjoint operator J (the fundamental symmetry) with $JJ^*=I$, $J=J^*$ such that

$$[x,y]=(Jx,y) \quad \text{for } x,y\in\mathcal{H}, \tag{2}$$

see, e.g., [2,5,14].

Conversely, every bounded and boundedly invertible selfadjoint operator G in a Hilbert space $(\mathcal{H},(\cdot,-))$ defines an inner product via

$$[\cdot,-]:=(G\cdot,-) \tag{3}$$

and $(\mathcal{H},[\cdot,-])$ becomes a Krein space. In particular, if the spectrum of G consists on the positive (or negative) semiaxis only of finitely many isolated eigenvalues of finite multiplicity, then $(\mathcal{H},[\cdot,-])$ is a Pontryagin space.

Eq. (3) is the starting point for various generalizations. E.g., if G is a bounded selfadjoint operator (but no more boundedly invertible) in $\mathcal{H}$ such that $\sigma(G)\cap(-\infty,\varepsilon)$ consists of finitely many eigenvalues of G with finite multiplicities for some $\varepsilon>0$, then $(\mathcal{H},[\cdot,-])$, where $[\cdot,-]$ is defined by (3), is called an Almost Pontryagin space, see [9]. Observe that in this case zero is allowed to be an eigenvalue of G with finite multiplicity. Almost Pontryagin spaces and operators therein were considered in various situations, we mention only [1,9–12,17,21,26,27]. The more general case that G is a bounded selfadjoint operator in $\mathcal{H}$ such that zero is an isolated eigenvalue of G with finite multiplicity gives rise to Almost Krein spaces, see [3]. Spaces with an inner product given by an arbitrary bounded selfadjoint operator were studied, e.g., in [16,22]. For applications we refer to [4,6,8–12,15,17–20,26,27].

In all the above-mentioned generalizations of (3) the selfadjointness of the operator G in $\mathcal{H}$ is maintained and the bounded invertibility is dropped. Obviously, this is the same as generalizing (2) by dropping $JJ^*=I$ and preserving $J=J^*$. From this point of view, it seems natural to generalize (2) the other way: dropping selfadjointness and preserving unitarity of J. The inner product space $(\mathcal{H},[\cdot,-])$, where $[\cdot,-]$ is defined by (2) with a unitary operator J is called an *S-space*, cf. [23] and also Definition 2.1 below. Moreover, the pair $((\cdot,-),J)$ is called a *Hilbert space realization* of the S-space $(\mathcal{H},[\cdot,-])$. Evidently, by definition every Krein space is a special case of an S-space.

In this paper we continue the study of S-spaces and operators therein started in [23,24]. It is known from [24] that the inner products of two Hilbert space realizations $((\cdot,-)_1,U_1)$ and $((\cdot,-)_2,U_2)$ define the same topology. Here, we show in particular that U_1 and U_2 are similar operators with respect to this topology, cf. Proposition 2.4. In Section 3 we introduce the notion of selfadjoint operators in S-spaces. We show that their spectrum is symmetric with respect to the real axis. As a main result we prove that to each selfadjoint operator A in an S-space $(\mathfrak{S},[\cdot,-])$ we find an inner product $\langle\cdot,-\rangle$ on $\mathfrak{S}$ such that $(\mathfrak{S},\langle\cdot,-\rangle)$ is a Krein space with the same topology as $(\mathfrak{S},[\cdot,-])$ and A is a selfadjoint operator in the Krein space $(\mathfrak{S},\langle\cdot,-\rangle)$, cf. Theorem 3.13.

Moreover, if $((\cdot,-),U)$ is a Hilbert space realization, we show in Theorem 3.13 below that each spectral subspace of U related to a Borel subset Δ of the unit circle which is symmetric with respect to the origin (i.e. $x\in\Delta$ implies $-x\in\Delta$) is invariant under A. Hence, in this paper we obtain the rather unexpected result: Each selfadjoint operator in an S-space is a selfadjoint operator in a Krein space with many invariant subspaces, provided the spectrum of the operator U from some Hilbert space realization $((\cdot,-),U)$ of $(\mathfrak{S},[\cdot,-])$ is sufficiently rich, i.e., if it consists of more than two points.

2. Definition and basic properties

The following definition is taken from [23].

Definition 2.1. A complex linear space $\mathfrak{S}$ with an inner product $[\cdot,-]$, that is a mapping from $\mathfrak{S}\times\mathfrak{S}$ into $\mathbb{C}$ which is linear in the first variable and conjugate linear in the other, is said to be an *S-space* if there is a Hilbert space structure in $\mathfrak{S}$ given by a positive definite inner product $(\cdot,-)$ and if there is a unitary operator U in the Hilbert space $(\mathfrak{S},(\cdot,-))$ such that

$$[f,g]=(Uf,g)\quad \text{for all } f,g\in\mathfrak{S}.$$

We refer to $[\cdot,-]$ as the inner product of $\mathfrak{S}$. The pair $((\cdot,-),U)$ is called a *Hilbert space realization* of $(\mathfrak{S},[\cdot,-])$.

Note, that the inner product $[\cdot,-]$ is not Hermitian, in general. An S-space is a Krein space if and only if the operator U in Definition 2.1 is in addition selfadjoint in the Hilbert space $(\mathfrak{S},(\cdot,-))$. For the theory of operators in Krein spaces we refer to [2,5].

Proposition 2.2. *Let $\mathfrak{S}$ be a complex linear space with an inner product $[\cdot,-]$. Then the pair $(\mathfrak{S},[\cdot,-])$ is an S-space if and only if there exist a Hilbert space inner product $(\cdot,-)$ on $\mathfrak{S}$ and a bounded and boundedly invertible normal operator T in $(\mathfrak{S},(\cdot,-))$ such that*

$$[f,g]=(Tf,g)\quad \textit{for all } f,g\in\mathfrak{S}.$$

Proof. We define the operator $U:=T(T^*T)^{-1/2}$ and the inner product

$$\langle x,y\rangle := \big((T^*T)^{1/2}x,y\big),\quad x,y\in\mathfrak{S}.$$

Since T is bijective, this is a Hilbert space inner product on $\mathfrak{S}$. From the relation $(T(T^*T)^{-1/2}T^*)^2=TT^*$ it follows that

$$\big(TT^*\big)^{1/2}=\big(T^*T\big)^{1/2}=T\big(T^*T\big)^{-1/2}T^*. \tag{4}$$

Hence, for $x,y\in\mathfrak{S}$ we obtain

$$\langle Ux,y\rangle=\big(\big(T^*T\big)^{1/2}T\big(T^*T\big)^{-1/2}x,y\big)=\big(T\big(T^*T\big)^{-1/2}T^*T\big(T^*T\big)^{-1/2}x,y\big)=[x,y]$$

and

$$\begin{aligned}\langle Ux,Uy\rangle &= \big(\big(T^*T\big)^{1/2}T\big(T^*T\big)^{-1/2}x,T\big(T^*T\big)^{-1/2}y\big)\\ &= \big(T\big(T^*T\big)^{-1/2}T^*T\big(T^*T\big)^{-1/2}x,T\big(T^*T\big)^{-1/2}y\big)\\ &= \big(Tx,T\big(T^*T\big)^{-1/2}y\big)=\big(\big(T^*T\big)^{-1/2}T^*Tx,y\big)\\ &= \big(\big(T^*T\big)^{1/2}x,y\big)=\langle x,y\rangle,\end{aligned}$$

which shows that U is unitary in $(\mathfrak{S},\langle\cdot,\cdot\rangle)$ and $(\mathfrak{S},[\cdot,-])$ is an S-space. □

Lemma 2.3. *Let $(\mathfrak{S}, [\cdot\,, -])$ be an S-space. Then there exists a uniquely defined linear operator $D : \mathfrak{S} \to \mathfrak{S}$ such that*

$$[x, y] = \overline{[y, Dx]} \quad \textit{for all } x, y \in \mathfrak{S}. \tag{5}$$

If $((\cdot\,, -), U)$ is a Hilbert space realization of $(\mathfrak{S}, [\cdot\,, -])$, then $D = U^2$.

Proof. Let $((\cdot\,, -), U)$ be a Hilbert space realization of $(\mathfrak{S}, [\cdot\,, -])$. Then it is easily seen that U^2 satisfies the relation (5) (with D replaced by U^2). Let $D : \mathfrak{S} \to \mathfrak{S}$ be a linear operator satisfying (5). Then from $[y, Dx] = [y, U^2x]$ for all $x, y \in \mathfrak{S}$ we conclude $(Uy, Dx - U^2x) = 0$ for all $x, y \in \mathfrak{S}$. And since U is bijective, it follows that $D = U^2$. □

The topology of an S-space $(\mathfrak{S}, [\cdot\,, -])$ is given by the topology induced by the Hilbert space inner product $(\cdot\,, -)$ of some Hilbert space realization of $(\mathfrak{S}, [\cdot\,, -])$. The following proposition states in particular that it does not depend on the choice of the Hilbert space realization, see also [24].

Proposition 2.4. *Let $(\mathfrak{S}, [\cdot\,, -])$ be an S-space and assume that there are two Hilbert space realizations $((\cdot\,, -)_1, U_1)$ and $((\cdot\,, -)_2, U_2)$ with*

$$[f, g] = (U_1 f, g)_1 = (U_2 f, g)_2 \quad \textit{for all } f, g \in \mathfrak{S}.$$

Then $(\cdot\,, -)_1$ and $(\cdot\,, -)_2$ are equivalent and the Gram operator S, defined by

$$(f, g)_2 = (Sf, g)_1 \quad \textit{for } f, g \in \mathfrak{S},$$

is bounded, boundedly invertible and selfadjoint with respect to $(\cdot\,, -)_1$ and with respect to $(\cdot\,, -)_2$. Moreover, the following statements hold:

(i) $U_1^2 = U_2^2$.
(ii) *The spectral measures of S in $(\mathfrak{S}, (\cdot\,, -)_1)$ and $(\mathfrak{S}, (\cdot\,, -)_2)$ coincide and we have*

$$S = U_1 U_2^{-1} = U_1^{-1} U_2, \quad \textit{and} \quad U_1^{-1} S U_1 = S^{-1} = U_2^{-1} S U_2. \tag{6}$$

Hence, the operator S is unitarily equivalent to its inverse.

(iii) *The operators U_1 and U_2 are similar. We have*

$$U_1 = S^{1/2} U_2 S^{-1/2}.$$

Hence

$$\sigma(U_1) = \sigma(U_2).$$

Proof. Denote by $\|\cdot\|_1$ and $\|\cdot\|_2$ the norms induced by $(\cdot\,, -)_1$ and $(\cdot\,, -)_2$, respectively, and set $B_1 := \{y \in \mathfrak{S}\colon\ \|y\|_1 = 1\}$. Then, for $y \in B_1$ the linear functional

$$F_y := [\cdot\,, y] = (U_1 \cdot\,, y)_1 = (U_2 \cdot\,, y)_2$$

is continuous on both $(\mathfrak{S}, (\cdot\,,-)_1)$ and $(\mathfrak{S}, (\cdot\,,-)_2)$. For its corresponding operator norms $\|F_y\|_{\mathcal{L}((\mathfrak{S},(\cdot\,,-)_1),\mathbb{C})}$ and $\|F_y\|_{\mathcal{L}((\mathfrak{S},(\cdot\,,-)_2),\mathbb{C})}$, respectively, we obtain $\|F_y\|_{\mathcal{L}((\mathfrak{S},(\cdot\,,-)_1),\mathbb{C})} = 1$ and $\|F_y\|_{\mathcal{L}((\mathfrak{S},(\cdot\,,-)_2),\mathbb{C})} = \|y\|_2$. For all $x \in \mathfrak{S}$ we have $\sup_{y\in B_1} |F_y(x)| \leqslant \|x\|_1 < \infty$. Due to the principle of uniform boundedness there exists some $c \in (0, \infty)$ with

$$\sup_{y\in B_1} \|F_y\|_{\mathcal{L}((\mathfrak{S},(\cdot\,,-)_2),\mathbb{C})} \leqslant c.$$

This yields $\|y\|_2 \leqslant c\|y\|_1$ for all $y \in \mathfrak{S}$. By interchanging the roles of $\|\cdot\|_1$ and $\|\cdot\|_2$ we obtain that these two norms are equivalent. Hence, by the well-known Lax–Milgram Theorem there exists a unique bounded linear operator S, selfadjoint in $(\mathfrak{S}, (\cdot\,,-)_1)$, such that

$$(f, g)_2 = (Sf, g)_1 \quad \text{for } f, g \in \mathfrak{S}.$$

It is boundedly invertible since $\|Sf_n\|_1 \to 0$ and $\|f_n\|_1 = 1$ would imply $\|f_n\|_2^2 = (Sf_n, f_n)_1 \to 0$ which contradicts the above proven fact that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. For $f, g \in \mathfrak{S}$ we have

$$(Sf, g)_2 = \big(S^2 f, g\big)_1 = (Sf, Sg)_1 = (f, Sg)_2.$$

Thus, S is also selfadjoint with respect to $(\cdot\,,-)_2$. Moreover, as $(\cdot\,,-)_1$ and $(\cdot\,,-)_2$ are positive definite, the operator S is uniformly positive.

Now we will show (i)–(iii). Statement (i) follows directly from Lemma 2.3. The equality of the spectral measures E_1 and E_2 of S in $(\mathfrak{S}, (\cdot\,,-)_1)$ and $(\mathfrak{S}, (\cdot\,,-)_2)$ follows from the equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ and Stone's formula (see, e.g., [7, XII.2]),

$$\begin{aligned} E_1\big((a,b)\big) &= \lim_{\delta\to 0+} \lim_{\epsilon\to 0+} \frac{1}{2\pi \mathrm{i}} \int_{a+\delta}^{b-\delta} \big(S - (\lambda + \epsilon \mathrm{i})\big)^{-1} - \big(S - (\lambda - \epsilon \mathrm{i})\big)^{-1}\, d\lambda \\ &= E_2\big((a,b)\big), \end{aligned} \tag{7}$$

where the limit is taken in the strong operator topology. As

$$(Sf, g)_1 = (f, g)_2 = \big(U_2 U_2^{-1} f, g\big)_2 = \big[U_2^{-1} f, g\big] = \big(U_1 U_2^{-1} f, g\big)_1,$$

we have $S = U_1 U_2^{-1}$ and, with (i), we conclude $S = U_1^{-1} U_1^2 U_2^{-1} = U_1^{-1} U_2$. We will denote the adjoint with respect to $(\cdot\,,-)_1$ by the symbol $*_1$ and the adjoint with respect to $(\cdot\,,-)_2$ by $*_2$. For $f, g \in \mathfrak{S}$ we have

$$\begin{aligned} (U_2 f, g)_2 = (SU_2 f, g)_1 = (U_2 f, Sg)_1 &= \big(f, U_2^{*_1} Sg\big)_1 \\ &= \big(S^{-1} f, U_2^{*_1} Sg\big)_2 = \big(f, S^{-1} U_2^{*_1} Sg\big)_2, \end{aligned}$$

thus

$$U_2^{*_2} = S^{-1} U_2^{*_1} S. \tag{8}$$

This implies

$$S = S^{*_1} = \left(U_1 U_2^{-1}\right)^{*_1} = \left(U_2^{-1}\right)^{*_1} U_1^{-1} = \left(S U_2^{*_2} S^{-1}\right)^{-1} U_1^{-1} = S U_2 S^{-1} U_1^{-1},$$

hence, with $S = U_1 U_2^{-1}$ we get $S^{-1} = U_1^{-1} S U_1$. Replacing U_1 by U_2 and U_2 by U_1 also $S^{-1} = U_2^{-1} S U_2$ holds and formula (6) and (ii) are proved.

By (ii) the square root of S in $(\mathfrak{S}, (\cdot\,,-)_1)$ and in $(\mathfrak{S}, (\cdot\,,-)_2)$ coincide. We denote the unique positive square root of the operator S by $S^{1/2}$. Since, by (6), $(U_1^{-1} S^{-1/2} U_1)^2 = U_1^{-1} S^{-1} U_1 = S$, we have the relation

$$S^{1/2} = U_1^{-1} S^{-1/2} U_1,$$

which yields

$$S^{1/2} U_2 S^{-1/2} = S^{1/2} S^{-1} U_1 S^{-1/2} = S^{-1/2} U_1 S^{-1/2} = U_1$$

and (iii) is proved. □

3. Linear operators in S-spaces

For the rest of this paper let $(\mathfrak{S}, [\cdot\,,-])$ be an S-space and let $((\cdot\,,-), U)$ be a fixed Hilbert space realization of $(\mathfrak{S}, [\cdot\,,-])$. In the following all topological notions are related to the Hilbert space topology given by $(\cdot\,,-)$. Its topology is independent of the particular choice of a Hilbert space realization (see Proposition 2.4).

Let T be a densely defined operator in a Hilbert space with a Hilbert space inner product $(\cdot\,,-)$. As usual, we denote by T^* the adjoint of T with respect to $(\cdot\,,-)$. As T is densely defined, T^* is unique. If T is, in addition, a closed operator, then T^* is densely defined, see, e.g., [13, Theorem III, §5.5].

Definition 3.1. Let A be a closed, densely defined operator in an S-space. An adjoint $A^\natural$ with respect to $[\cdot\,,-]$ is defined via the following relations:

$$\operatorname{dom} A^\natural := \left\{g \in \mathfrak{S}\colon \exists h \in \mathfrak{S} \text{ with } [Af, g] = [f, h] \text{ for all } f \in \operatorname{dom} A\right\},$$
$$[Af, g] = \left[f, A^\natural g\right] \quad \text{for all } f \in \operatorname{dom} A \text{ and } g \in \operatorname{dom} A^\natural.$$

Analogously, we define ${}^\natural A$ via

$$\operatorname{dom} {}^\natural A := \left\{f \in \mathfrak{S}\colon \exists h \in \mathfrak{S} \text{ with } [f, Ag] = [h, g] \text{ for all } g \in \operatorname{dom} A\right\},$$
$$[f, Ag] = \left[{}^\natural A f, g\right] \quad \text{for all } g \in \operatorname{dom} A \text{ and } f \in \operatorname{dom} {}^\natural A.$$

In the following proposition (see [24]) we collect some of the properties of $A^\natural$ and ${}^\natural A$. We provide here a short proof in order to make this exposition self-contained.

Proposition 3.2. *The operators $A^\natural$ and ${}^\natural A$ are closed, densely defined and satisfy*

$$\operatorname{dom} A^\natural = U \operatorname{dom} A^* = \operatorname{dom}\left(A^* U^*\right) \quad \textit{and} \quad A^\natural = U A^* U^* \tag{9}$$

and

$$\operatorname{dom} {}^\natural A = U^* \operatorname{dom} A^* = \operatorname{dom}\bigl(A^*U\bigr) \quad \textit{and} \quad {}^\natural A = U^*A^*U. \tag{10}$$

Proof. Obviously, we have $f \in \operatorname{dom}(A^*U^*)$ if and only if $U^*f \in \operatorname{dom} A^*$ which in turn holds if and only if $f \in U \operatorname{dom} A^*$. Hence $U \operatorname{dom} A^* = \operatorname{dom}(A^*U^*)$.

Let $g \in \operatorname{dom} A^\natural$. By Definition 3.1 we have for all $f \in \operatorname{dom} A$

$$\bigl(f, U^*A^\natural g\bigr) = \bigl[f, A^\natural g\bigr] = [Af, g] = \bigl(Af, U^*g\bigr).$$

Thus $U^*g \in \operatorname{dom} A^*$ and $U^*A^\natural \subset A^*U^*$.

If $g \in \operatorname{dom}(A^*U^*)$, then we have for all $f \in \operatorname{dom} A$

$$\bigl[f, UA^*U^*g\bigr] = \bigl(f, A^*U^*g\bigr) = \bigl(Af, U^*g\bigr) = [Af, g].$$

Hence $g \in \operatorname{dom} A^\natural$ and $A^\natural \subset UA^*U^*$. This gives $U^*A^\natural = A^*U^*$ and (9) is proved. The proof of (10) is similar and we omit it here. □

Recall that for a densely defined operator T and a bounded operator X in a Hilbert space we have (see [25, Section 4.4])

$$(XT)^* = T^*X^* \quad \text{and, if } X \text{ is boundedly invertible,} \quad (TX)^* = X^*T^*. \tag{11}$$

Proposition 3.3. *If ${}^\natural A = A^\natural$ then $AD = DA$ where $D = U^2$.*

Proof. If ${}^\natural A = A^\natural$, then from Proposition 3.2 and (11) we conclude

$${}^\natural\bigl(A^\natural\bigr) = {}^\natural\bigl(UA^*U^*\bigr) = U^*\bigl(UA^*U^*\bigr)^*U = A,$$

and hence, with ${}^\natural A = A^\natural$,

$$A = {}^{\natural\natural}A = U^*\bigl({}^\natural A\bigr)^*U = U^*\bigl(U^*A^*U\bigr)^*U = \bigl(U^*\bigr)^2AU^2 = D^*AD.$$

And since D is unitary, the assertion follows. □

Corollary 3.4. *If ${}^\natural A = A^\natural$ and U has no eigenvalues, then A does not have eigenvalues with finite geometric multiplicity.*

Proof. By Proposition 3.3 we have $AD = DA$. Assume that λ is an eigenvalue of A with finite geometric multiplicity. From $AD = DA$ it follows that $\ker(A - \lambda)$ is invariant under D. Therefore, D (and hence U) has eigenvalues. □

Definition 3.5. A densely defined operator A in the S-space $(\mathfrak{S}, [\,\cdot\,, -])$ is called *selfadjoint* if

$$A = A^\natural.$$

We have the following characterization for selfadjointness of operators in S-spaces.

Proposition 3.6. *For a densely defined operator A in $\mathfrak{S}$ the following assertions are equivalent*:

(i) $A = A^\natural$, *i.e., A is selfadjoint in* $(\mathfrak{S}, [\,\cdot\,, -])$.
(ii) $U^*A = A^*U^*$.
(iii) $UA = A^*U$.
(iv) $A = {}^\natural A$.

If one of these equivalent statements holds true we have

$$f \in \operatorname{dom} A \quad \Longleftrightarrow \quad U^* f \in \operatorname{dom} A^* \quad \Longleftrightarrow \quad Uf \in \operatorname{dom} A^*. \tag{12}$$

Proof. The equivalence of (i) and (ii) follows from (9), the equivalence of (iii) and (iv) follows from (10).

Assume that (ii) holds. For $f \in \operatorname{dom} A$ we conclude $U^* f \in \operatorname{dom} A^*$. This implies for $f, g \in \operatorname{dom} A$:

$$(f, UAg) = \big(A^*U^*f, g\big) = \big(U^*Af, g\big) = (Af, Ug)$$

and we have $Ug \in \operatorname{dom} A^*$, hence $UA \subset A^*U$. For the other inclusion, we observe by (ii) that $\operatorname{dom} A^* = U^* \operatorname{dom} A$. For $Ug \in \operatorname{dom} A^*$ and $f \in \operatorname{dom} A$ we have $U^*f \in \operatorname{dom} A^*$ and

$$\big(U^*f, U^*A^*Ug\big) = \big(f, A^*Ug\big) = (Af, Ug) = \big(U^*Af, g\big) = \big(A^*U^*f, g\big),$$

thus $g \in \operatorname{dom}(A^*)^* = \operatorname{dom} A$. This gives $U^*A^*Ug = Ag$ and $A^*U \subset UA$. This proves (iii).

Assume that (iii) holds. For $f \in \operatorname{dom} A$ we conclude $Uf \in \operatorname{dom} A^*$. This gives for $f, g \in \operatorname{dom} A$

$$\big(U^*Ag, f\big) = (Ag, Uf) = \big(g, A^*Uf\big) = (g, UAf) = \big(U^*g, Af\big)$$

and we have $U^*g \in \operatorname{dom} A^*$, hence $U^*A \subset A^*U^*$. For the other inclusion, we observe by (iii) that $\operatorname{dom} A^* = U \operatorname{dom} A$. For $U^*g \in \operatorname{dom} A^*$ and $f \in \operatorname{dom} A$ we have $Uf \in \operatorname{dom} A^*$ and

$$\big(Uf, UA^*U^*g\big) = \big(f, A^*U^*g\big) = \big(Af, U^*g\big) = (UAf, g) = \big(A^*Uf, g\big),$$

thus $g \in \operatorname{dom}(A^*)^* = \operatorname{dom} A$. This gives $A^*U^*g = U^*Ag$ and $A^*U^* \subset U^*A$. This proves (ii). Moreover, we have shown that (12) holds. $\square$

Proposition 3.7. *Let A be a selfadjoint operator in the S-space* $(\mathfrak{S}, [\,\cdot\,, -])$. *Then the spectrum of A is symmetric with respect to the real axis.*

Proof. Since $A = A^\natural = UA^*U^*$, cf. Proposition 3.2, the operator A is unitarily equivalent to its adjoint. Hence, $\sigma(A) = \sigma(A^*) = \{\bar{\lambda}\colon \lambda \in \sigma(A)\}$. $\square$

Let A be a selfadjoint operator in the S-space $(\mathfrak{S}, [\,\cdot\,, -])$. If $(\mathfrak{S}, [\,\cdot\,, -])$ is a Krein space, then U is selfadjoint and thus $\sigma(U) = \sigma_p(U) \subset \{-1, 1\}$. It is well known that the spectrum of A may be rather arbitrary. For example, it can happen that $\sigma(A) = \mathbb{C}$.

Example 3.8. Assume that – in contrast to the Krein space case – $\sigma(U)$ consists of two eigenvalues λ_1, λ_2 with $\lambda_1 \neq -\lambda_2$, e.g., $\sigma(U) = \{1, \mathrm{i}\}$. Then $\sigma(U^2) = \{1, -1\}$, and since A commutes with $D = U^2$ by Proposition 3.3 the spectral subspaces of D are A-invariant. Since these coincide with the eigenspaces of U corresponding to 1 and i, respectively, we have $A = A_1 \oplus A_{\mathrm{i}}$ and $U = I \oplus \mathrm{i}I$ with respect to the decomposition $\mathfrak{S} = \ker(U-1) \oplus \ker(U-\mathrm{i})$. From the selfadjointness of A in $(\mathfrak{S}, [\,\cdot\,,-])$ we conclude that both A_1 and A_{i} are selfadjoint with respect to the Hilbert space scalar product $(\,\cdot\,,-)$ in $\ker(U-1)$ and $\ker(U-\mathrm{i})$, respectively. Hence, A is selfadjoint in $(\mathfrak{S}, (\,\cdot\,,-))$. In particular its spectrum is real.

This simple example shows that it is not necessarily "better" to know that an operator is selfadjoint in a Krein space than in an S-space. In fact, we will show in the following that every selfadjoint operator in an S-space is also selfadjoint in some Krein space. However, in general (if $\sigma(U) \neq \{\mathrm{e}^{\mathrm{i}t}, -\mathrm{e}^{\mathrm{i}t}\}$ for some $t \in [0,\pi)$) the selfadjointness in the S-space gives us more information about the operator. E.g., we automatically know a whole bunch of invariant subspaces of the operator – namely the spectral subspaces of D.

Definition 3.9. Let G be a bounded selfadjoint operator in the Hilbert space $(\mathfrak{S}, (\,\cdot\,,-))$. A closed and densely defined linear operator T in $\mathfrak{S}$ will be called *G-symmetric* if $GT \subset (GT)^*$. The operator T is called *G-selfadjoint* if $GT = (GT)^*$.

In the following we will deal with the operators

$$G(t) := \frac{1}{2i}\big(\mathrm{e}^{\mathrm{i}t}U - \mathrm{e}^{-\mathrm{i}t}U^*\big), \quad t \in [0,\pi).$$

It is easily seen that all these operators are bounded selfadjoint operators in the Hilbert space $(\mathfrak{S}, (\,\cdot\,,-))$. We have $G(0) = \operatorname{Im} U$ and $G(\pi/2) = \operatorname{Re} U$. Moreover, the operator $G(t)$ can be factorized in the following way

$$G(t) = \frac{\mathrm{e}^{\mathrm{i}t}}{2i}U^*\big(U^2 - \mathrm{e}^{-2\mathrm{i}t}\big) = \frac{\mathrm{e}^{\mathrm{i}t}}{2i}U^*\big(U - \mathrm{e}^{-\mathrm{i}t}\big)\big(U + \mathrm{e}^{-\mathrm{i}t}\big).$$

Therefore, $G(t)$ is boundedly invertible if and only if $\mathrm{e}^{-\mathrm{i}t}, -\mathrm{e}^{-\mathrm{i}t} \in \rho(U)$. In this case $(\mathfrak{S}, (G(t)\,\cdot\,,-))$ is a Krein space.

Proposition 3.10. *Let A be a selfadjoint operator in the S-space $(\mathfrak{S}, [\,\cdot\,,-])$. Then A is $G(t)$-symmetric for all $t \in [0,\pi)$. If for some $t \in [0,\pi)$ we have $\mathrm{e}^{-\mathrm{i}t}, -\mathrm{e}^{-\mathrm{i}t} \in \rho(U)$, then the operator A is $G(t)$-selfadjoint.*

Proof. Let $t \in [0,\pi)$. Then by Proposition 3.6 we have

$$\begin{aligned} G(t)A &= \frac{1}{2i}\big(\mathrm{e}^{\mathrm{i}t}U - \mathrm{e}^{-\mathrm{i}t}U^*\big)A = \frac{1}{2i}\big(\mathrm{e}^{\mathrm{i}t}UA - \mathrm{e}^{-\mathrm{i}t}U^*A\big) \\ &= \frac{1}{2i}\big(\mathrm{e}^{\mathrm{i}t}A^*U - \mathrm{e}^{-\mathrm{i}t}A^*U^*\big) \\ &\subset A^*G(t) = \big(G(t)A\big)^*. \end{aligned}$$

This shows that A is $G(t)$-symmetric.

We have by Proposition 3.3 $AD = DA$, therefore for each complex number λ

$$(D-\lambda)A \subset A(D-\lambda). \tag{13}$$

We will show that for $\lambda \in \rho(D)$ equality holds,

$$(D-\lambda)A = A(D-\lambda). \tag{14}$$

Let $\lambda \in \rho(D)$. We have to show $\operatorname{dom}(A(D-\lambda)) \subset \operatorname{dom} A$. Consider the Hilbert space $\mathfrak{S}_A := (\operatorname{dom} A, (\cdot\,,-)_A)$, where the inner product $(\cdot\,,-)_A$ is defined by

$$(f,g)_A := (f,g) + (Af, Ag), \quad f,g \in \operatorname{dom} A.$$

Due to $AD = DA$ the linear manifold $\operatorname{dom} A$ is D-invariant. Hence, define

$$D_A : \mathfrak{S}_A \to \mathfrak{S}_A, \qquad D_A f := Df, \quad f \in \operatorname{dom} A.$$

For $f, g \in \mathfrak{S}_A$ we have

$$(D_A f, D_A g)_A = (Df, Dg) + (ADf, ADg) = (f,g) + (DAf, DAg) = (f,g)_A$$

and D_A is an isometric operator in $\mathfrak{S}_A$. Assume that there exists $z \in \mathfrak{S}_A$ with $(D_A f, z)_A = 0$ for all $f \in \mathfrak{S}_A$. That gives

$$-\bigl(f, D^*z\bigr) = (DAf, Az) = \bigl(Af, D^*Az\bigr)$$

for all $f \in \mathfrak{S}_A$ and, hence, $D^*Az \in \operatorname{dom} A^*$ with $A^*D^*Az = -D^*z$. By (11) and $AD = DA$ we obtain

$$-D^*z = (DA)^*Az = (AD)^*Az = D^*A^*Az.$$

It follows $A^*Az = -z$ and $0 \leqslant (A^*Az, z) = -(z,z) \leqslant 0$. Therefore $z = 0$ and D_A has a dense range in $\mathfrak{S}_A$. The operator D_A is a unitary operator in $\mathfrak{S}_A$.

For $\lambda \in \rho(D) \setminus \{0\}$, we have

$$\operatorname{ran}(D_A - \lambda)^{\perp_A} = \ker\bigl(D_A^{-1} - \bar{\lambda}\bigr) = \ker\bigl(D_A^{-1}\bar{\lambda}\bigl(\overline{\lambda^{-1}} - D_A\bigr)\bigr) = \{0\},$$

where $\perp_A$ denotes the orthogonal complement in $\mathfrak{S}_A$ with respect to $(\cdot\,,-)_A$. Hence, for $\lambda \in \rho(D)$, the operator $D_A - \lambda$ has a dense range in $\mathfrak{S}_A$.

In order to show (14) let $f \in \operatorname{dom}(A(D-\lambda))$. Then $(D-\lambda)f \in \operatorname{dom} A$. As $\operatorname{ran}(D_A - \lambda)$ is dense in $\mathfrak{S}_A$, there exists a sequence (f_n) in $\operatorname{dom} A$ such that

$$\bigl\|(D_A - \lambda)f_n - (D-\lambda)f\bigr\|_A \to 0 \quad \text{as } n \to \infty.$$

From this we conclude

$$f_n \to f \quad \text{and} \quad A(D-\lambda)f_n \to A(D-\lambda)f \quad \text{as } n \to \infty$$

(in $\mathfrak{S}$). But $f_n \in \operatorname{dom} A$ and from (13) it follows that

$$f_n \to f \quad \text{and} \quad Af_n \to (D-\lambda)^{-1}A(D-\lambda)f \quad \text{as } n \to \infty.$$

Now, it is a consequence of the closedness of A that $f \in \operatorname{dom} A$ and $(D-\lambda)Af = A(D-\lambda)f$. This shows (14).

The selfadjointness of A in $(\mathfrak{S}, [\cdot, -])$ is equivalent to $A^*U^* = U^*A$, cf. Proposition 3.6. With $\pm\mathrm{e}^{-\mathrm{i}t} \in \rho(U)$ we have $\mathrm{e}^{-2\mathrm{i}t} \in \rho(D)$. This and (14) yield

$$\begin{aligned} A^*G(t) &= \frac{\mathrm{e}^{\mathrm{i}t}}{2\mathrm{i}}A^*U^*\big(D-\mathrm{e}^{-2\mathrm{i}t}\big) = \frac{\mathrm{e}^{\mathrm{i}t}}{2\mathrm{i}}U^*A\big(D-\mathrm{e}^{-2\mathrm{i}t}\big) \\ &= \frac{\mathrm{e}^{\mathrm{i}t}}{2\mathrm{i}}U^*\big(D-\mathrm{e}^{-2\mathrm{i}t}\big)A = G(t)A, \end{aligned}$$

which is the $G(t)$-selfadjointness of A. $\square$

Note that in general the operator A in Proposition 3.11 is not $G(t)$-selfadjoint. For example let $U := iI$ and suppose that A is unbounded. Then $G(\pi/2) = 0$ and $G(\pi/2)A$ is the restriction of the zero operator to $\operatorname{dom} A$, whereas $(G(\pi/2)A)^*$ equals the zero operator on $\mathfrak{S}$. Hence, in this case, A is not $G(\pi/2)$-selfadjoint.

If $G(t)$ is boundedly invertible, then the space $\mathfrak{S}$ equipped with the inner product $(G(t)\cdot, -)$ is a Krein space. The following theorem follows immediately from Proposition 3.10.

Theorem 3.11. *Let A be a selfadjoint operator in the S-space $(\mathfrak{S}, [\cdot, -])$. If for some $t \in [0, \pi)$ we have $\mathrm{e}^{-\mathrm{i}t}, -\mathrm{e}^{-\mathrm{i}t} \in \rho(U)$, then the operator A is selfadjoint in the Krein space $(\mathfrak{S}, (G(t)\cdot, -))$.*

If in the situation of Theorem 3.11 the operator U satisfies some additional assumptions, more can be said about the spectrum of A.

Theorem 3.12. *Let A be a selfadjoint operator in the S-space $(\mathfrak{S}, [\cdot, -])$ and assume that there is some $t \in [0, \pi)$ such that $\mathrm{e}^{-\mathrm{i}t}, -\mathrm{e}^{-\mathrm{i}t} \in \rho(U)$. Let $\mathbb{T} = \mathbb{T}_1 \dot\cup \mathbb{T}_2$ be a decomposition of the unit circle, where*

$$\mathbb{T}_1 := \big\{\mathrm{e}^{\mathrm{i}s}\colon -t \leqslant s < -t+\pi\big\} \quad \text{and} \quad \mathbb{T}_2 := \big\{\mathrm{e}^{\mathrm{i}s}\colon -t+\pi \leqslant s < -t+2\pi\big\}.$$

If $\mathbb{T}_1 \cap \sigma(U) = \emptyset$ or $\mathbb{T}_2 \cap \sigma(U) = \emptyset$ then A is selfadjoint in the Hilbert space $(\mathfrak{S}, (G(t)\cdot, -))$. In particular,

$$\sigma(A) \subset \mathbb{R}.$$

If $\mathbb{T}_1 \cap \sigma(U)$ or $\mathbb{T}_2 \cap \sigma(U)$ consists of finitely many κ isolated eigenvalues (counted with multiplicity) of U, then the non-real spectrum of A in the open upper half-plane consists of at most κ isolated eigenvalues with finite algebraic multiplicities (counted with multiplicity),

$$\sigma(A) \setminus \mathbb{R} = \{\lambda_1, \overline{\lambda_1}, \lambda_2, \overline{\lambda_2}, \ldots, \lambda_{\kappa_0}, \overline{\lambda_{\kappa_0}}\} \subset \sigma_p(A),$$

for some κ_0 with $0 \leqslant \kappa_0 \leqslant \kappa$.

Proof. We define

$$\widetilde{U} := \mathrm{e}^{\mathrm{i}t} U.$$

Then $\pm 1 \in \rho(\widetilde{U})$. The operator A is selfadjoint in the S-space $(\mathfrak{S}, [\,\cdot\,, -]_\sim)$, where $[\,\cdot\,, -]_\sim$ is given by

$$[f, g]_\sim := (\widetilde{U} f, g) \quad \text{for all } f, g \in \mathfrak{S}.$$

By Theorem 3.11, A is selfadjoint in the Krein space $(\mathfrak{S}, (\operatorname{Im}\widetilde{U}\cdot\,, \cdot))$. If $\mathbb{T}_1 \cap \sigma(U) = \emptyset$ then $\operatorname{Im}\widetilde{U}$ is a uniformly negative operator in the Hilbert space $(\mathfrak{S}, (\,\cdot\,, -))$, and hence A is a self-adjoint operator in the Hilbert space $(\mathfrak{S}, -(\operatorname{Im}\widetilde{U}\cdot\,, \cdot))$. A similar argument holds for the case $\mathbb{T}_2 \cap \sigma(U) = \emptyset$ and the first assertion of the theorem is proved.

If $\mathbb{T}_1 \cap \sigma(U)$ consists of finitely many isolated eigenvalues of U with finite multiplicity then $\operatorname{Im}\widetilde{U}$ is a bounded and boundedly invertible selfadjoint operator in the Hilbert space $(\mathfrak{S}, (\,\cdot\,, -))$. Moreover, the spectral subspace of $\operatorname{Im}\widetilde{U}$ corresponding to the positive real numbers is finite dimensional. Therefore A is a selfadjoint operator in the Pontryagin space $(\mathfrak{S}, (\operatorname{Im}\widetilde{U}\cdot\,, \cdot))$ and the second assertion of the theorem follows from well-known properties of selfadjoint operators in Pontryagin spaces, see, e.g., [2,5]. Similar arguments apply if $\mathbb{T}_2 \cap \sigma(U)$ consists of finitely many isolated eigenvalues of U. $\square$

The following theorem is the main result of this paper. It shows that the notions of S-space selfadjointness and Krein space selfadjointness coincide.

Theorem 3.13. *Let A be a selfadjoint operator in the S-space $(\mathfrak{S}, [\,\cdot\,, -])$. Then there exists a Krein space inner product $\langle\,\cdot\,, -\rangle$ such that A is selfadjoint in the Krein space $(\mathfrak{S}, \langle\,\cdot\,, -\rangle)$. Moreover, if E_U denotes the spectral measure of U and if Δ is a Borel subset of the unit circle $\mathbb{T}$ with the property that $\lambda \in \Delta$ implies $-\lambda \in \Delta$, then the spectral subspace $E_U(\Delta)\mathfrak{S}$ is an invariant subspace for A.*

Proof. We choose some $\varepsilon \in (0, \pi/2)$ and define

$$\Delta_1 := \{\mathrm{e}^{\mathrm{i}t}\colon t \in (-\varepsilon, \varepsilon)\} \cup \{-\mathrm{e}^{\mathrm{i}t}\colon t \in (-\varepsilon, \varepsilon)\}, \quad \Delta_2 := \mathbb{T} \setminus \Delta_1.$$

Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be the spectral subspaces of U corresponding to Δ_1 and Δ_2, respectively, i.e.

$$\mathfrak{S}_1 = E_U(\Delta_1)\mathfrak{S} \quad \text{and} \quad \mathfrak{S}_2 = E_U(\Delta_2)\mathfrak{S}.$$

Then we have

$$\mathfrak{S} = \mathfrak{S}_1 \oplus \mathfrak{S}_2.$$

We define the sets

$$\Delta_1^2 := \{\mathrm{e}^{\mathrm{i}t}\colon t \in (-2\varepsilon, 2\varepsilon)\} \quad \text{and} \quad \Delta_2^2 := \mathbb{T} \setminus \Delta_1^2 = \{z^2\colon z \in \Delta_2\}.$$

If E_{U^2} denotes the spectral measure of U^2 and $h : \mathbb{C} \to \mathbb{C}$ denotes the function given by $h(z) = z^2$, then we deduce from the properties of the functional calculus for unitary operators for $j = 1, 2$

$$E_{U^2}(\Delta_j^2) = \mathbf{1}_{\Delta_j^2}(U^2) = (\mathbf{1}_{\Delta_j^2} \circ h)(U) = \mathbf{1}_{h^{-1}(\Delta_j^2)}(U) = E_U(\Delta_j),$$

where $\mathbf{1}_\Delta$ is the indicator function corresponding to a Borel set Δ and $h^{-1}(\Delta_j^2)$ denotes the preimage of Δ_j^2 under h. Therefore, the spectral subspace of $D = U^2$ corresponding to Δ_j^2 coincides with $\mathfrak{S}_j$, $j = 1, 2$.

For $\lambda \in \rho(D)$ the operator $(D - \lambda)^{-1}$ commutes with A, cf. (14). With some obvious modifications due to the fact that U is a unitary operator, the projector $E_U(\Delta_j)$, $j = 1, 2$, can be written in a similar form as in (7). From this, we conclude

$$E_U(\Delta_j)A \subset AE_U(\Delta_j). \tag{15}$$

Hence, for $x \in \operatorname{dom} A$ we have $E_U(\Delta_j)x \in \operatorname{dom} A$ and

$$\operatorname{dom} A = (\mathfrak{S}_1 \cap \operatorname{dom} A) \oplus (\mathfrak{S}_2 \cap \operatorname{dom} A).$$

Moreover, if $x \in \mathfrak{S}_j \cap \operatorname{dom} A$ then with (15)

$$Ax = E_U(\Delta_j)Ax,$$

which implies that the subspaces $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are A-invariant. Thus, with respect to the decomposition $\mathfrak{S} = \mathfrak{S}_1 \oplus \mathfrak{S}_2$ the operators A and U decompose as $A = A_1 \oplus A_2$ and $U = U_1 \oplus U_2$, where $A_j = A|\mathfrak{S}_j$ and $U_j = U|\mathfrak{S}_j$, $j = 1, 2$. It is easy to see that A_1 is selfadjoint in the S-space $(\mathfrak{S}_1, (U_1 \cdot, -))$ and that A_2 is selfadjoint in the S-space $(\mathfrak{S}_2, (U_2 \cdot, -))$. Since $i, -i \in \rho(U_1)$ and $1, -1 \in \rho(U_2)$, it follows from Theorem 3.11 that there are Krein space inner products $\langle \cdot, - \rangle_1$ and $\langle \cdot, - \rangle_2$ in $\mathfrak{S}_1$ and $\mathfrak{S}_2$, respectively, such that A_j is selfadjoint in the Krein space $(\mathfrak{S}_j, \langle \cdot, - \rangle_j)$, $j = 1, 2$. Hence, A is obviously selfadjoint in the Krein space $(\mathfrak{S}, \langle \cdot, - \rangle)$, where $\langle \cdot, - \rangle$ is given by

$$\langle x, v \rangle := \langle x_1, y_1 \rangle_1 + \langle x_2, y_2 \rangle_2,$$

$x = x_1 + x_2$, $y = y_1 + y_2$, $x_1, x_2 \in \mathfrak{S}_1$, $x_2, y_2 \in \mathfrak{S}_2$. $\square$

Remark 3.14. Each Krein space is also an S-space, hence, obviously, every selfadjoint operator in a Krein space is simultaneously selfadjoint in an S-space. Theorem 3.13 shows that in a sense the contrary is true as well. Hence, for each selfadjoint operator A in an S-space $\mathfrak{S}$ we find an inner product which turns $\mathfrak{S}$ into a Krein space and A into a selfadjoint operator with respect to this inner product. In addition, as revealed in the proof of Theorem 3.13, all spectral subspaces of U which correspond to Borel sets symmetric with respect to $z \mapsto -z$ are invariant subspaces of A.

Example 3.15. As an illustration of Theorem 3.13 we consider a simple example with 2×2 matrices. Let U be unitary in $\mathbb{C}^2$ and choose an orthonormal basis of $\mathbb{C}^2$ such that the corresponding matrix is diagonal with entries $z_1, z_2 \in \mathbb{T}$. A matrix with entries $a, b, c, d \in \mathbb{C}$ which is selfadjoint in the S-space given by U has to satisfy

$$\begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix} \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix},$$

cf. Proposition 3.6, part (iii). We assume $cb \neq 0$. From this we see that a and d are real, $z_1 = \pm z_2$ and $b = \pm \bar{c}$. Hence, either the matrix is selfadjoint (in the case $z_1 = z_2$) or, if $z_1 = -z_2$, we have $b = -\bar{c}$ and the matrix is selfadjoint in the (finite dimensional) Krein space with fundamental symmetry

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

4. Concluding remarks

S-spaces are Hilbert spaces with an additional inner product given by a unitary Gramian U. Krein spaces are special cases of S-spaces as their Gramian can be chosen to be selfadjoint and simultaneously unitary.

From this point of view, the class of S-spaces is larger than the class of Krein spaces. It is the main result of this paper that the class of selfadjoint operators in S-spaces is not larger than the corresponding class in Krein spaces.

Moreover, Theorem 3.13 reveals an interesting fact: A selfadjoint operator in an S-space having a Gramian U with spectrum larger than the set $\{-1, 1\}$ has invariant subspaces – a fact which is not known a priori for selfadjoint operators in Krein spaces. An interesting, and so far unanswered, question is: Which class of selfadjoint operators in Krein spaces are selfadjoint in an S-space with a Gramian U which has a spectrum larger than $\{-1, 1\}$ (and, hence, gives rise to many invariant subspaces)?

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